CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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CIP seminar, Friday conversations:,
For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks.

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory.
4. MRS factorisation: A local system of coordinates for Hausdorff groups and fine tuning between analysis and algebra.
5. This scope is a continent and a long route, let us, today, walk part of the way together.

Disclaimer. — The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

CCRT[24] On the rôle of local analysis in the computation of polylogarithms and harmonic sums II.

1. In the preceding weeks, we have considered the MRS factorization which is one of our precious jewels.

$$\mathcal{D}_X := \sum_{w\in X^*} w \otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \tag{1}$$

2. Last week, we have seen how to extend the indexation of Polylogarithmic functions and Harmonic sums.
3. But Polylogarithmic functions are ruled out by shuffles and Harmonic sums by stuffle or Hadamard products.
4. We must have a tool to state identity (1) in the context of stuffle products or, more generally, deformed shuffle products (this deformation is, indeed, a perturbation).

Introduction.

5. We have explained, firstly how to extend polylogarithms

$$\mathrm{Li}(s_1,\ldots s_r)=\sum_{n_1>n_2>\ldots n_r>0}\frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}\ \text{for } |z|<1 \tag{2}$$

They were a priori coded by lists $(s_1,\ldots s_r)$ but, when $s_i\in\mathbb{N}_+$, they admit an *iterated integral representation* and are better coded by words with letters in $X=\{x_0,x_1\}$. We will use the one-to-one correspondences.

$$(s_1,\ldots,s_r)\in\mathbb{N}_+^r\leftrightarrow x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1\in X^*x_1\leftrightarrow y_{s_1}\ldots y_{s_r}\in Y^* \tag{3}$$

- $\mathrm{Li}(s)[z]$ is Jonquière and, for $\Re(s)>1$, one has $\mathrm{Li}(s)[1]=\zeta(s)$
- Completed by $Li(x_0^n)=\frac{log^n(z)}{n!}$ this provides a family of $\mathbb{C}$-independant functions (linearly) admitting an analytic continuation on the cleft plane $\mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$ or $\widetilde{\mathbb{C}\setminus\{0,1\}}$.

Introduction: Recap of the facts.

6. Starting from $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ $(\Re(s) > 1)$
7. and the multiplication of two of these

$$\zeta(s_1)\zeta(s_2) = \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} = \zeta(s_1, s_2) + \zeta(s_1 + s_2) + \zeta(s_2, s_1)$$

8. then several of them, then mixing this with classical polylogarithms defined, for $k \geq 1, |z| < 1$, by

$$-\log(1-z) = \mathrm{Li}_1 = \sum_{n\geq 1} \frac{z^n}{n^1};\ \mathrm{Li}_2 = \sum_{n\geq 1} \frac{z^n}{n^2};\ \ldots;\ \mathrm{Li}_k(z) := \sum_{n\geq 1} \frac{z^n}{n^k}$$

9. We obtained quantities called polylogarithms

$$Li_{y_{s_1}\ldots y_{s_k}}(z) := \sum_{n_1 > \ldots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \ldots n_k^{s_k}}\ ;\ |z| < 1$$

Introduction: Recap of the facts/2

- They satisfy the recursion (ladder stepdown)

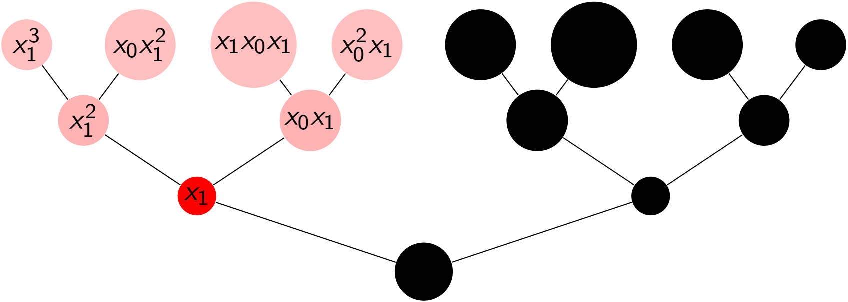
$$\begin{aligned} z\frac{d}{dz}Li_{y_{s_1}\dots y_{s_k}} &= Li_{y_{s_1-1}\dots y_{s_k}} \text{ if } s_1>1 \\ (1-z)\frac{d}{dz}Li_{y_1 y_{s_2}\dots y_{s_k}} &= Li_{y_{s_2}\dots y_{s_k}} \text{ if } k>1 \end{aligned} \tag{4}$$

which, with $s_i \in \mathbb{N}_{\geq 1},\ k \geq 1$, ends at the "seed"

$$\mathrm{Li}_{y_1}(z) = \mathrm{Li}_1(z) = \log(\frac{1}{1-z}) \tag{5}$$

- For the next step, we code the moves $z\frac{d}{dz}$ (resp. $(1-z)\frac{d}{dz}$) - or more precisely sections $\int_0^z \frac{f(s)}{s}ds$ (resp. $\int_0^z \frac{f(s)}{1-s}ds$) - with x_0 (resp. x_1).

Tree of outputs (so far).



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $*_0 = 0$

$$\langle S \mid x_1^n \rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S \mid x_0 x_1 \rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0 x_1}(z) = \sum_{n \geq 1} \frac{z^n}{n^2}$$

$$\langle S \mid x_0^2 x_1 \rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2 x_1}(z) = \sum_{n \geq 1} \frac{z^n}{n^3} \quad ; \quad \langle S \mid x_1 x_0 x_1 \rangle = \mathrm{Li}_{x_1 x_0 x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}}{n_1 n_2^2}$$

$$\langle S \mid x_0 x_1^2 \rangle = \mathrm{Li}_{x_0 x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}}{n_1^2 n_2} \quad ;$$

above "cl. not." stands for "classical notation"

Introduction: Review of the facts/3

- Calling S the prospective generating series

$$S = \sum_{w\in X^*} \underbrace{\langle S \mid w\rangle}_{\in\mathcal{H}(\Omega)} w \ ; \ X = \{x_0, x_1\} \tag{6}$$

V. Drinfel'd [1] indirectly proposed a way to complete the tree:

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S & (NCDE) \\ \lim_{\substack{z\to 0\\ z\in\Omega}} S(z)e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} & (Asympt.\ Init.\ Cond.) \end{cases} \tag{7}$$

from the general theory, this system has a unique solution which is precisely Li (called G_0 in [1]) ; $S \mapsto \mathbf{d}(S)$ being the term by term derivation of the coefficients.

- Minh [2] indicated a way to effectively compute this solution through (improper) iterated integrals (see also [13]).

Explicit construction of Drinfeld's G_0.

Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

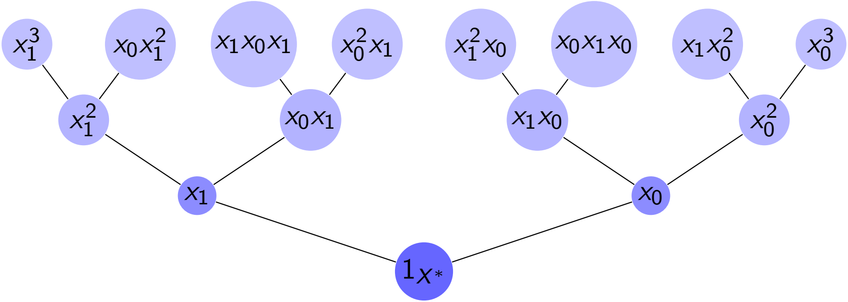
$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \alpha_0^s(u)\frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \alpha_0^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \ (w \in x_0^*) \\ \int_0^z \alpha_0^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \ (w \in x_0 X^* x_1 x_0^*) \end{cases}$$

The third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$ is Li (G_0 in [1]).

Complete tree of outputs.



As an example, we compute some coefficients

$$\langle \mathrm{Li} \mid x_0^n \rangle = \frac{log(z)^n}{n!} \quad ; \quad \langle \mathrm{Li} \mid x_1^n \rangle = \frac{(-log(1-z))^n}{n!}$$

$$\langle \mathrm{Li} \mid x_0x_1 \rangle = \mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2} \quad ; \quad \langle \mathrm{Li} \mid x_1x_0 \rangle = \langle \mathrm{Li} \mid x_1 \shuffle x_0 - x_0x_1 \rangle (z)$$

$$\langle \mathrm{Li} \mid x_0^2x_1 \rangle = \mathrm{Li}_3(z) = \sum_{n\geq 1} \frac{z^n}{n^3} \quad ; \quad \langle \mathrm{Li} \mid x_1x_0 \rangle = (-log(1-z))log(z) - \mathrm{Li}_2(z)$$

$$\langle \mathrm{Li} \mid x_0^{r-1}x_1 \rangle = \mathrm{Li}_r(z) = \sum_{n\geq 1} \frac{z^n}{n^r} \quad ; \quad \langle \mathrm{Li} \mid x_1^2x_0 \rangle = \langle \mathrm{Li} \mid \frac{1}{2}(x_1 \shuffle x_1 \shuffle x_0) - (x_1 \shuffle x_0x_1) + x_0x_1^2 \rangle$$

Li From a NCDE.

The generating series $S = \sum_{w \in X^*} Li(w)$ satisfies (and is unique to do so)

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z \to 0 \\ z \in \Omega}} S(z) e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} \end{cases} \tag{8}$$

with $X = \{x_0, x_1\}$. This is, up to the sign of x_1, the solution G_0 of Drinfel'd [13] for KZ3[a]. We define this unique solution as Li. All Li_w are $\mathbb{C}$- and even $\mathbb{C}(z)$-linearly independant (see CAP 17 *Linear independance without monodromy* [24]).

[a] In fact, the path from KZ3 to these equations is done through a counter-homogenization (see Vu's forthcoming talks).

Domain of Li (global, definition)

In order to extend indexation of Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S=\sum_{n\geq 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geq 0} Li_{S_n}(z)$ converges unconditionally for compact convergence in Ω. One sets

$$Li_S(z) := \sum_{n\geq 0} Li_{S_n}(z) \tag{9}$$

Starting the ladder

$$\begin{tikzcd}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) \arrow[r, hook, two heads, "\mathrm{Li}_\bullet"] \arrow[d, hook] & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \arrow[d, hook] \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] \arrow[r, two heads, "\mathrm{Li}_\bullet^{(1)}"] & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{tikzcd}$$

Examples

$$Li_{x_0^*}(z) = z,\quad Li_{x_1^*}(z) = (1-z)^{-1},\ Li_{\alpha x_0^* + \beta x_1^*}(z) = z^{\alpha}(1-z)^{-\beta}$$

Main difference between $\alpha_{z_0}^z$ and α_0^z.

6. Here, we still work with $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ and $u_0 = 1/z,\ u_1 = 1/(1-z)$
7. $\alpha_{z_0}^z,\ \alpha_0^z :\ X^* \longrightarrow \mathcal{H}(\Omega)$ are both shuffle characters (see below) but they satisfy different growth conditions.
8. **With $\alpha_{z_0}^z$, ($z_0 \in \Omega$). —** Let us denote $\mathfrak{K}(\Omega)$ the set of compact subsets of Ω. One can show that, for all $K \in \mathfrak{K}(\Omega)$, there exists $M_K > 0$ s.t.
$$(\forall w \in X^+)(\ ||\langle \alpha_{z_0}^z \mid w\rangle||_K \leq M_K \frac{1}{(|w|-1)!}\ ) \tag{10}$$
9. This entails that, given a rational series $T = \sum_{n\geq 0} T_n$ (where $T_n = \sum_{|w|=n} \langle T \mid w\rangle$), the series, for all $K \in \mathfrak{K}(\Omega)$
$$\sum_{n\geq 0} ||\langle \alpha_{z_0}^z \mid T_n\rangle||_K < +\infty$$
10. We will say that $T \in Dom(\alpha_{z_0}^z)$ and set $\alpha_{z_0}^z(T) = \sum_{n\geq 0}\langle \alpha_{z_0}^z \mid T_n\rangle$.

Main difference between $\alpha_{z_0}^z$ and $\alpha_0^z/2$

11. In fact, α_0^z satisfies no condition of the type (10) because, with $x_0^* x_1$ (Jonquière branch), we can see that
    1. for $n \geq 1$, $(x_0^* x_1)_n = x_0^{n-1} x_1$, then
    $$\langle \mathrm{Li}(z) \mid x_0^{n-1} x_1 \rangle = \langle \alpha_{z_0}^z \mid x_0^{n-1} x_1 \rangle = J_n(z) = \sum_{k \geq 1} \frac{z^k}{k^n} \tag{11}$$
    2. The series $\sum_{n \geq 0} J_n$ does not converge (even pointwise) on $]0,1[$ because,
    $$x \in]0,1[\Longrightarrow J_n(x) \geq x$$
    3. So, what can be salvaged ? $\rightarrow$ in fact, conditions (growth or other) implying absolute convergence at the level of words is hopeless because of restriction and we would like to preserve
    $$\mathrm{Li}(x_0^*) = z \ ; \ \mathrm{Li}(x_1^*) = 1/(1-z) \ ; \ \mathrm{Li}(S ш T) = \mathrm{Li}(S).\mathrm{Li}(T) \tag{12}$$
    and then $\mathrm{Li}\left((x_0 + x_1)^*\right) = z/(1-z)$

Main difference between $\alpha_{z_0}^z$ and $\alpha_0^z/3$

12. Then, we must have a criterium (for admitting a series in $Dom(\mathrm{Li})$)
13. Fortunately $\mathcal{H}(\Omega)$ shares with finite dimensional spaces the following property

$$\text{Unconditional convergence} \Longleftrightarrow \text{Absolute convergence} \tag{13}$$

14. **Unconditional convergence** for a series $\sum_{n\geq 0} u_n$ means convergence "independent of the order" i.e. that $\sum_{n\geq 0} u_{\sigma(n)}$ converges whatever $\sigma \in \mathfrak{S}_{\mathbb{N}}$.
15. **Absolute convergence** is wrt the continuous seminorms of the space.
16. Time is ripe now to speak of the standard topology of $\mathcal{H}(\Omega)$.
17. For $K \in \mathfrak{K}(\Omega)$, we introduce the seminorm (norm if Ω is connected and $K^\circ \neq \emptyset$)

$$||f||_K = \sup_{z\in K} |f(z)|$$

Initial topologies.

18. We now use a very very general construction, well suited both for series and holomorphic functions (and many other situations), that of initial topologies (see [34] and, for a detailed construction [6], Ch1 §2.3)

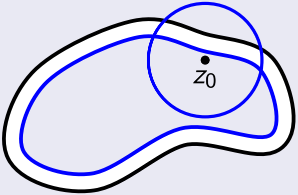
$$
\begin{tikzcd}
 & & \mathscr{C}(K_i;\mathbb{C}) \arrow[d, dash, dotted] \\
X \arrow[r, "g"'] \arrow[urr, "res_{K_i}\circ g"] & \mathcal{H}(\Omega) \arrow[ur, "res_{K_i}"'] \arrow[r, "res_{K_j}"] \arrow[dr, "res_{K_r}"'] & \mathscr{C}(K_j;\mathbb{C}) \arrow[d, dash, dotted] \\
 & & \mathscr{C}(K_r;\mathbb{C})
\end{tikzcd}
$$

19. So $\mathcal{H}(\Omega)$ is a locally convex TVS whose topology is defined by the family of seminorms $(|| \; ||_K)_{K\in\mathfrak{K}(\Omega)}$.

Topology of $\mathcal{H}(\Omega)$ cont'd.

18 In fact, every $\Omega \subset \mathbb{C}$ is σ-compact, this means that one can construct a sequence $(K_n)_{n\geq 1}$ of compacts i.e. $(\forall K \in \mathfrak{K}(\Omega))(\exists n \geq 1)(K \subset K_n)$ therefore $\mathcal{H}(\Omega)$ is a complete (hence closed) subset of the product $\Pi_{n\geq 1}\,\mathcal{C}(K_n;\mathbb{C})$ (for the topology on the cube, see a next CCRT).

$$K_n = \{z \in \Omega \mid d(z,z_0) \leq n \text{ and } d(z,\mathbb{C}\setminus\Omega) \geq \frac{1}{n}\}.$$



19 We will see more (step-by-step and starting from scratch) on the topology of the cube and separability in the CCRT devoted to convergence questions).

Domain of Li.

18. If $\Omega \neq \emptyset$, $\mathcal{H}(\Omega)$ is not normable because, there are two continuous operators
$$a^\dagger : \ f \mapsto z.f \ ; \ a : \ f \mapsto \frac{d}{dz} f$$
such that $[a, a^\dagger] = Id_{\mathcal{H}(\Omega)}$ (**Hint** Compute $ad_a(e^{ta^\dagger})$).
19. $\mathcal{H}(\Omega)$ has property (13) (nuclearity).
20. This leads us to the following

Definition

Let $T \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle$, we define (with $[S]_n := \sum_{|w|=n} \langle S \mid w\rangle\, w$)
$$Dom(T) = \{S \in \mathbb{C}\langle\langle X\rangle\rangle \mid \sum_{n\geq 0} \langle T \mid \ [S]_n\rangle \text{ cv inconditionally}\} \tag{14}$$
If $S \in Dom(T)$, we set $\langle T \mid S\rangle := \sum_{n\geq 0} \langle T \mid \ [S]_n\rangle$.

Shuffle properties and domain of Li.

18. In the case when T is a shuffle character, we have

Theorem (GD, Quoc Huan Ngô, HNM [14] for Li)

Let $T \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ *such that*

$$\langle T \mid : \ P \mapsto \langle T \mid P\rangle \ (\mathbb{C}\langle X\rangle \to \mathcal{H}(\Omega)) \tag{15}$$

is a shuffle character. then
i) Dom(T) *is a shuffle subalgebra of* $(\mathbb{C}\langle\langle X\rangle\rangle, ш, 1_{X^*})$.
ii) $\langle T \mid S_1 ш S_2\rangle = \langle T \mid S_1\rangle\langle T \mid S_2\rangle$ *i.e.* $S \mapsto \langle T \mid S\rangle$ *is a shuffle character of* $(Dom(T), ш, 1_{X^*})$ *that we will still denote* $\langle T \mid$.
iii) Then Im$(\langle T \mid\ )$ *is a (unital) subalgebra of* $\mathcal{H}(\Omega)$.
iv) In particular (see **infra** *for an algebraic proof),* $z = \mathrm{Li}(x_0^*)$ *and then,* $\mathbb{C}[z] \subset Im(Dom(\mathrm{Li}))$.

Open problems and some solved.

19. Do we have $\mathcal{H}(\Omega) = \overline{Im(Dom(\mathrm{Li}))}$ $(= \overline{Im(\mathrm{Li})})$? (in other words does it exist inaccessible $f \in \mathcal{H}(\Omega)$?)
20. If $z_0 \notin \Omega$, does $1/(z - z_0)$ belong to $Im(\mathrm{Li})$? ($z_0 \in \overline{\Omega}$ and $z_0 \notin \overline{\Omega}$)
21. (Solved) Are there non-rational series in $Dom(\mathrm{Li})$? (answer **yes**)
22. (Solved) Is $\mathbb{C}^{rat}\langle\langle X\rangle\rangle$ contained in $Dom(\mathrm{Li})$ (answer **no**)
23. What is the topological complexity of $Dom(\mathrm{Li})$ in the **Borel hierarchy** (Addison notations, see [25] for details and use the convenient framework of polish spaces [7], ch IX).
24. **Borel hierarchy**: We recall that this hierarchy is indexed by ordinals and defined as follows
    1. A set is in $\mathbf{\Sigma}^0_1$ if and only if it is open.
    2. A set is in $\mathbf{\Pi}^0_\alpha$ if and only if its complement is in $\mathbf{\Sigma}^0_\alpha$.
    3. A set A is in $\mathbf{\Sigma}^0_\alpha$ for $\alpha > 1$ if and only if there is a sequence of sets $A_1, A_2, \ldots$ such that each A_i is in $\mathbf{\Pi}^0_{\alpha_i}$ for some $\alpha_i < \alpha$ and $A = \bigcup A_i$.
    4. A set is in $\mathbf{\Delta}^0_\alpha$ if and only if it is both in $\mathbf{\Sigma}^0_\alpha$ and in $\mathbf{\Pi}^0_\alpha$.

Open problems and some solved/2

25. From slide (11), one can remark that the iterated integrals are based on two integrators, informally defined as

$$\iota_1(f) := \int_0^z f(s)\frac{ds}{1-s} \ ; \ \iota_0(f) := \int_{z_0}^z f(s)\frac{ds}{s} \text{ with } z_0 \in \{0,1\} \tag{16}$$

ι_1 is defined and continous on $\mathcal{H}(\Omega)$ and ι_0 is defined on $span_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}$ [a] (context-dependent) and not continuous [14] on this set (see below).
Problem What is the Baire class of ι_0 ?

26. Recall that $\mathfrak{K}(\Omega)$ admits a cofinal sequence $(K_n)_{n\in\mathbb{N}}$ of compacts i.e. $(\forall K \in \mathfrak{K}(\Omega))(\exists n \in \mathbb{N})(K \subset K_n)$ therefore $\mathcal{H}(\Omega)$ is a complete (hence closed) subset of the product $\Pi_{n\in\mathbb{N}}\mathcal{C}(K_n;\mathbb{C})$.

27. Recall that (see [14] and slide SI.18)

$$K_n = \{z \in \Omega \mid d(z, z_0) \leq n \text{ and } d(z, \mathbb{C}\setminus\Omega) \geq \frac{1}{n}\}.$$

[a] It can be a little bit extended, see our paper [14].

Properties.

Proposition

With this definition, we have

1. $Dom(Li)$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and so is
$$Dom^{rat}(Li) := Dom(Li) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$$
2. For $S, T \in Dom(Li)$, we have
$$\mathrm{Li}_{S \,ш\, T} = \mathrm{Li}_S \,.\, \mathrm{Li}_T$$

Examples and counterexamples

For $|t| < 1$, one has $(tx_0)^* x_1 \in Dom(Li, D)$ (D being the open unit slit disc and $Dom(Li, D)$ defined similarly), whereas $x_0^* x_1 \notin Dom(Li, D)$. Indeed, we have to examine the convergence of $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z)$, but, for $z \in]0, 1[$, one has $0 < z < \mathrm{Li}_{x_0^n x_1}(z) \in \mathbb{R}$ and therefore, for these values $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z) = +\infty$. Contrariwise one can show that, for $|t| < 1$,
$$\mathrm{Li}_{(tx_0)^* x_1}(z) = \sum_{n\geq 1} \frac{z^n}{n-t}$$

Passing to harmonic sums $\mathrm{H}_w,\ w \in Y^*$.

Polylogarithms having a removable singularity at zero

The following proposition helps us characterize their indices.

Proposition

Let $f(z) = \langle \mathrm{Li} \mid P \rangle = \sum_{w \in X^*} \langle P \mid w \rangle \, \mathrm{Li}_w$. *The following conditions are equivalent*

i) f *can be analytically extended around zero*

ii) $P \in \mathbb{C}\langle X \rangle x_1 \oplus \mathbb{C}.1_{X^*}$

We recall the expansion (for $w \in X^* x_1 \sqcup \{1_{X^*}\},\ |z| < 1$)

$$\frac{\mathrm{Li}_w(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{\pi_Y(w)}(N) \, z^N \tag{17}$$

Global and local domains.

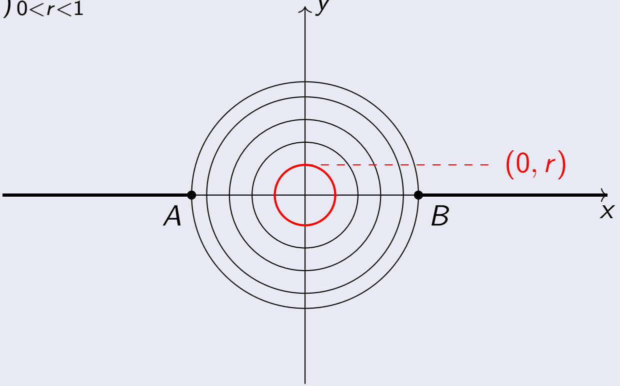
This proposition and the lemma lead us to the following definitions.

1. *Global domains*.–
   Let $\emptyset \neq \Omega \subset \widetilde{B}$ (with $B = \mathbb{C} \setminus \{0,1\}$), we define $Dom_{\Omega}(Li) \subset \mathbb{C}\langle\langle X \rangle\rangle$ to be the set of series $S = \sum_{n \geq 0} S_n$ (with $S_n = \sum_{|w|=n} \langle S \mid w \rangle\, w$ each homogeneous component) such that $\sum_{n \in \mathbb{N}} Li_{S_n}$ is unconditionally convergent for the compact convergence (UCC) [27].
   As examples, we have Ω_1, the doubly cleft plane then $Dom(\mathrm{Li}) := Dom_{\Omega_1}(\mathrm{Li})$ or $\Omega_2 = \widetilde{B}$
2. *Local domains around zero (fit with H-theory)*.–
   Here, we consider series $S \in (\mathbb{C}\langle\langle X \rangle\rangle x_1 \oplus \mathbb{C}\, 1_{X^*})$ (i.e. $supp(S) \cap X x_0 = \emptyset$). We consider radii $0 < R \leq 1$, the corresponding open discs $D_R = \{z \in \mathbb{C} |\ |z| < R\}$ and define

$$\begin{aligned} Dom_R(\mathrm{Li}) &:= \{S = \Sigma_{n \geq 0} S_n \in (\mathbb{C}\langle\langle X \rangle\rangle x_1 \oplus \mathbb{C} 1_{\Omega}) | \sum_{n \in \mathbb{N}} Li_{S_n} \text{ (UCC) in } D_R\} \\ Dom_{loc}(\mathrm{Li}) &:= \cup_{0 < R \leq 1} Dom_R(\mathrm{Li}). \end{aligned}$$

Local domains.

28. Local domains: the domain of convergence of $\mathrm{Li}_w,\ w \in X^*x_1$ is $\mathbb{C} \setminus (]-\infty,-1] \cup [1,+\infty[)$ and these functions are Taylor expandable around zero. With $S = \sum_{n\geq 0} S_n \in \mathbb{C}\langle\langle X \rangle\rangle$, we study the inconditional convergence of $\sum_{n\geq 0} \mathrm{Li}_{S_n}(z)$ within different open disks $(B_{(0,0)}(r))_{0<r<1}$



Properties of the domains.

Theorem A

1. For all $\emptyset \neq \Omega \subset \widetilde{B}$, $Dom_{\Omega}(\mathrm{Li})$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X \rangle\rangle$ and so are the $Dom_R(\mathrm{Li})$.
2. $R \mapsto Dom_R(\mathrm{Li})$ is strictly decreasing for $R \in]0,1]$.
3. All $Dom_R(\mathrm{Li})$ and $Dom_{loc}(\mathrm{Li})$ are shuffle subalgebras of $\mathbb{C}\langle\langle X \rangle\rangle$ and $\pi_Y(Dom_{loc}(\mathrm{Li}))$ is a stuffle subalgebra of $\mathbb{C}\langle\langle Y \rangle\rangle$.
4. Conversely, let $T(z) = \sum_{N \geq 0} a_N z^N$ be a Taylor series i.e. such that $\limsup_{N \to +\infty} |a_N|^{1/N} = B < +\infty$, then the series

$$S = \sum_{N \geq 0} a_N (-(-x_1)^+)^{Ш N} \tag{18}$$

is summable in $\mathbb{C}\langle\langle X \rangle\rangle$ (with sum in $\mathbb{C}\langle\langle x_1 \rangle\rangle$) and $S \in Dom_R(Li)$ with $R = \frac{1}{B+1}$ and $\mathrm{Li}_S = T(z)$.

Theorem A/2

5. Let $S \in Dom_R(\mathrm{Li})$ and $S = \sum_{n\geq 0} S_n$ (homogeneous decomposition), we define[a] $N \mapsto \mathrm{H}_{\pi_Y(S)}(N)$ by

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(S)}(N) z^N \ . \tag{19}$$

Moreover, for all $r \in]0, R[$, we have

$$\sum_{n,N\geq 0} |\mathrm{H}_{\pi_Y(S_n)} r^N| < +\infty, \tag{20}$$

in particular, for all $N \in \mathbb{N}$ the series (of complex numbers) $\sum_{n\geq 0} \mathrm{H}_{\pi_Y(S_n)}(N)$ converges absolutely to $\mathrm{H}_{\pi_Y(S)}(N)$.

[a] This definition is compatible with the old one when S is a polynomial.

Theorem A/3

6. Conversely, let $Q \in \mathbb{C}\langle\langle Y \rangle\rangle$ with $Q = \sum_{n\geq 0} Q_n$ (decomposition by weights), we suppose that it exists $r \in]0, 1]$ such that

$$\sum_{n,N\geq 0} |\mathrm{H}_{Q_n}(N) r^N| < +\infty \tag{21}$$

in particular, for all $N \in \mathbb{N}$, $\sum_{n\geq 0} \mathrm{H}_{Q_n}(N) = \ell(N) \in \mathbb{C}$ unconditionally.
Under such circumstances, $\pi_X(Q) \in Dom_r(\mathrm{Li})$ and, for all $|z| < r$

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \ell(N) z^N, \tag{22}$$

Insightful fathers.

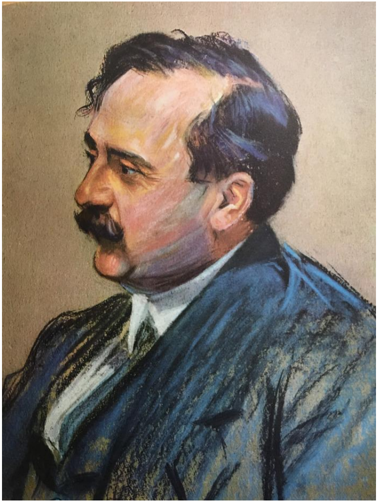


Figure: Jacques Hadamard and Paul Montel.

Local domains: morphism properties.

Corollary (of Theorem A)

Let $S, T \in \mathit{Dom}^{\mathrm{loc}}(\mathrm{Li})$*, then*

$$S \sqcup\!\sqcup\, T \in \mathit{Dom}^{\mathrm{loc}}(\mathrm{Li}), \pi_X(\pi_Y(S) \uplus \pi_Y(T)) \in \mathit{Dom}^{\mathrm{loc}}(\mathrm{Li})$$

and for all $N \geq 0$,

$$\begin{aligned} \mathrm{Li}_{S \sqcup\!\sqcup\, T} &= \mathrm{Li}_S \,\mathrm{Li}_T; \quad \mathrm{Li}_{1_{X^*}} = 1_{\mathcal{H}(\Omega)}, && (23)\\ \mathrm{H}_{\pi_Y(S) \uplus \pi_Y(T)}(N) &= \mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N). && (24)\\ \frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} &= \frac{\mathrm{Li}_{\pi_X(\pi_Y(S) \uplus \pi_Y(T))}(z)}{1-z}. && (25) \end{aligned}$$

Continuing the ladder

$$
\begin{tikzcd}
(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*}) \arrow[r, hook, two heads, "\mathrm{Li}_\bullet"] \arrow[d, hook] & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \arrow[d, hook] \\
(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] \arrow[r, two heads, "\mathrm{Li}_\bullet^{(1)}"] \arrow[d, hook] & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \arrow[d, hook] \\
\mathbb{C}\langle X\rangle \shuffle \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \shuffle \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \arrow[r, "\mathrm{Li}_\bullet^{(2)}"] & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \arrow[u, two heads] \arrow[ur, dashed] &
\end{tikzcd}
$$

We have, after a theorem by Leopold Kronecker,

$$\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \left\{ \frac{P}{Q} \right\}_{\substack{P,Q\in\mathbb{C}[x] \\ Q(0)\neq 0}} \tag{26}$$

On the right: freeness without monodromy.

Theorem (Deneufchâtel, GHED,Minh & Solomon, 2011 [12])

Let $(\mathcal{A}, \partial)$ be a k-commutative associative differential algebra with unit and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $\partial(\mathcal{C}) \subset \mathcal{C}$). We suppose that $k = \ker(\partial)$ and that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \;;\; \langle S \mid 1 \rangle = 1 \text{ with } M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle \tag{27}$$

(i.e. M is a homogeneous series of degree 1)
The following conditions are equivalent :

1. *The family $(\langle S \mid w \rangle)_{w \in X^*}$ of coefficients of S is (linearly) free over $\mathcal{C}$.*
2. *The family of coefficients $(\langle S \mid x \rangle)_{x \in X \cup \{1_{X^*}\}}$ is (linearly) free over $\mathcal{C}$.*
3. *The family $(u_x)_{x \in X}$ is such that, for $f \in \mathcal{C}$ et $\alpha_x \in k$*

$$\partial(f) = \sum_{x \in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

A useful property.

mathoverflow

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Independence of characters with respect to polynomials

5

1

I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U} \to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+ = ker(\epsilon)$. We build the following filtrations $(N \geq 1)$

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \ldots . . \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

and

$$\mathcal{U}_N^* = \mathcal{U}_{N+1}^{\perp} = \{f \in \mathcal{U}^* | (\forall u \in \mathcal{U}_{N+1})(f(u) = 0)\} \qquad (2)$$

the first one is decreasing and the second one increasing. One shows easily that (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^* \diamond \mathcal{U}_q^* \subset \mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^* = \cup_{n\geq 1} \mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Now, we can state the

Theorem : The set of characters of $(\mathcal{U}, ., 1_\mathcal{U})$ is linearly free w.r.t. $\mathcal{U}_\infty^*$.

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Left and then right: the arrow $\mathrm{Li}_\bullet^{(1)}$.

Proposition

i. The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle^{\mathrm{rat}}, ш, 1_{X^*})$.

ii. $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is a free module over $\mathbb{C}\langle X\rangle$, the family $\{(x_0^*)^{ш k} ш (x_1^*)^{ш l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}\langle X\rangle$-basis of it.

iii. As a consequence, $\{w ш (x_0^*)^{ш k} ш (x_1^*)^{ш l}\}_{\substack{w\in X^* \\ (k,l)\in\mathbb{Z}\times\mathbb{N}}}$ is a $\mathbb{C}$-basis of it.

iv. $\mathrm{Li}_\bullet^{(1)}$ is the unique morphism from $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*]$ to $\mathcal{H}(\Omega)$ such that

$$x_0^* \to z,\ (-x_0)^* \to z^{-1} \text{ and } x_1^* \to (1-z)^{-1}$$

v. $\mathrm{Im}(\mathrm{Li}_\bullet^{(1)}) = \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}$.

vi. $\ker(\mathrm{Li}_\bullet^{(1)})$ is the (shuffle) ideal generated by $x_0^* ш x_1^* - x_1^* + 1_{X^*}$.

Other combinatorial instances of MRS factorisation.

Shuffle product governs Poly- and Hyper- logarithms, stuffle governs Harmonic functions and one can see that other forms of perturbated shuffles govern other types of special functions.
In combinatorics (and computer science), one often uses products[a] defined by recursions on words of the form

$$\begin{aligned} u \,ш_\varphi 1_{Y^*} &= 1_{Y^*} \,ш_\varphi u = u \text{ and} \\ au \,ш_\varphi bv &= a(u \,ш_\varphi bv) + b(au \,ш_\varphi v) + \varphi(a,b)(u \,ш_\varphi v) \end{aligned}$$

where $\varphi : R.X \otimes R.X \to R.X$ is some associative law.

[a] as shuffle, stuffle, infiltration, q-infiltration.

G.H.E. Duchamp, J.-Y. Enjalbert, H. N. Minh, C. Tollu, *The mechanics of shuffle products and their siblings*, Discrete Mathematics, **340** (Sep. 2017)

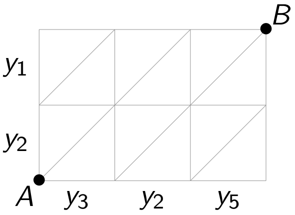
Examples of $\shuffle_\varphi$

Name	Formula (recursion)	φ	Type
Shuffle [21]	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$	$\varphi \equiv 0$	I
Stuffle [19]	$x_i u \uplus x_j v = x_i(u \uplus x_j v) + x_j(x_i u \uplus v) + x_{i+j}(u \uplus v)$	$\varphi(x_i, x_j) = x_{i+j}$	I
Min-stuffle [7]	$x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v) - x_{i+j}(u \sqcup v)$	$\varphi(x_i, x_j) = -x_{i+j}$	III
Muffle [14]	$x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v) + x_{i\times j}(u \sqcup v)$	$\varphi(x_i, x_j) = x_{i\times j}$	I
q-shuffle [3]	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	III
q-shuffle$_2$	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q^{i.j} x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	II
$\mathtt{LDIAG}(1, q_s)$ [10] (non-crossed, non-shifted)	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v) + q_s^{\|a\|\|b\|} a.b(u \shuffle v)$	$\varphi(a, b) = q_s^{\|a\|\|b\|}(a.b)$	II
q-Infiltration [12]	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b} a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b} a$	III
AC-stuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a, b)(u \shuffle_\varphi v)$	$\varphi(a, b) = \varphi(b, a)$ $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$	IV
Semigroup-stuffle	$x_t u \shuffle_\perp x_s v = x_t(u \shuffle_\perp x_s v) + x_s(x_t u \shuffle_\perp v) + x_{t\perp s}(u \shuffle_\perp v)$	$\varphi(x_t, x_s) = x_{t\perp s}$	I
φ-shuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a, b)(u \shuffle_\varphi v)$	$\varphi(a, b)$ law of AAU	V

Of course, the q-shuffle is equal to the (classical) shuffle when $q = 0$. As for the q-infiltration, when $q = 1$, one recovers the infiltration product defined in [6].

Many shuffle products arise in number theory when one studies polylogarithms, harmonic sums and polyzêtas: it was in order to study all these products that two of us introduced Type IV (see above) [15].

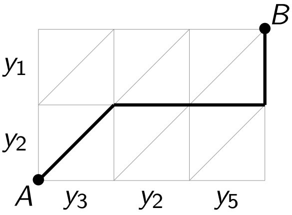
One can see the product $u \shuffle_{\varphi} v$ as a sum indexed by paths (with right-up-diagonal(ne) steps) within the grid formed by the two words (u horizontal and v vertical, the diagonal steps corresponding to the factors $\varphi(a, b)$)



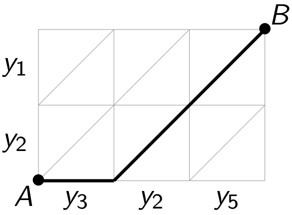
.

For example,

the path



reads $\color{red}{\varphi(y_3, y_2)}y_2 y_5 y_1$

the path 

reads $y_3{\color{red}\varphi(y_2, y_2)\varphi(y_5, y_1)}$. We have the following

Theorem (Radford theorem for $\shuffle_\varphi$)

Let R be a $\mathbb{Q}$-algebra (associative, commutative with unit) such that

$$\varphi : R\langle X\rangle \otimes R\langle X\rangle \to R\langle X\rangle$$

is associative.

If X is totally ordered by $<$, then $(\mathcal{L}yn(X)^{\shuffle_\varphi \alpha})_{\alpha\in\mathbb{N}^{(\mathcal{L}yn(X))}}$ is a linear basis of $R\langle X\rangle$.

In particular if, moreover, φ is commutative, then $(R\langle X\rangle, \shuffle_\varphi, 1_{X^*})$ is a polynomial algebra with $\mathcal{L}yn(X)$ as a transcendence basis.

Dualizability

If one considers φ as defined by its structure constants

$$\varphi(x,y) = \sum_{z \in X} \gamma_{x,y}^{z}\, z$$

one sees at once that $ш_\varphi$ is dualizable within $R\langle X\rangle$ iff the tensor $\gamma_{x,y}^{z}$ is locally finite in its contravariant place "z" i.e.

$$(\forall z \in X)(\#\{(x,y) \in X^2 | \gamma_{x,y}^{z} \neq 0\} < +\infty)\ .$$

Remark

Shuffle, stuffle, infiltration are dualizable. The comultiplication associated with Generalized Lerch Functions and $\top$ are not (see HNM's talk).

Dualizability/2

In case $\shuffle_{\varphi}$ is dualizable, one has a comultiplication

$$\Delta_{\shuffle_{\varphi}} : R\langle X\rangle \to R\langle X\rangle \otimes R\langle X\rangle$$

(with structure constants the transpose of the tensor $\gamma_{x,y}^{z}$). The following

$$\mathcal{B}_{\varphi}^{\vee} = (R\langle X\rangle, conc, 1_{X^*}, \Delta_{\shuffle_{\varphi}}, \varepsilon) \tag{28}$$

is a bialgebra in duality with $\mathcal{B}_{\varphi}$ (not always a Hopf algebra although the letter was so $\to$ ex. $\shuffle_{\varphi} = \uparrow_q$ i.e. the q-infiltration).

Associative commutative φ-deformed shuffle products

Theorem (CAP 2015)

Let us suppose that φ is associative and dualizable. We still denote the dual law of $\sqcup\!\sqcup_\varphi$ by $\Delta_{\sqcup\!\sqcup_\varphi} : R\langle Y\rangle \longrightarrow R\langle Y\rangle \otimes R\langle Y\rangle$, $\mathcal{B}_\varphi^\vee := (R\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\varphi}, \varepsilon)$ is a bialgebra. Moreover, if φ is commutative the following conditions are equivalent

i) $\mathcal{B}_\varphi^\vee$ *is an enveloping bialgebra.* (**CQMM** *theorem*)

ii) $\mathcal{B}_\varphi^\vee$ *is isomorphic to* $(R\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$ *as a bialgebra.*

iii) Δ_+ *is locally nilpotent (i.e. φ is moderate).*

iv) *For all $y \in Y$, the following series is a polynomial.*

$$\pi_1(y) = y + \sum_{l\geq 2} \frac{(-1)^{l-1}}{l} \sum_{x_1,\ldots,x_l \in Y} \langle y \mid \varphi(x_1 \ldots x_l)\rangle\, x_1 \ldots x_l.$$

In the previous equivalent cases, φ is called moderate.
In this case, one can straighten the $\sqcup\!\sqcup_\varphi$ product and imitate Lyndon basis computation in order to get a basis of the primitive elements and then have an effective calculus for Schützenberger factorisation.

Bialgebra structure

Theorem

Let R be a commutative ring (with unit). We suppose that the product φ is associative, then the algebra $(R\langle X\rangle, ш_{\varphi}, 1_{X^*})$ can be endowed with the comultiplication Δ_{conc} dual to the concatenation

$$\Delta_{conc}(w) = \sum_{uv=w} u \otimes v \tag{29}$$

and the "constant term" character $\varepsilon(P) = \langle P \mid 1_{X^*}\rangle$.

(i) *With this setting, we have a bialgebra [a].*

$$\mathcal{B}_{\varphi} = (R\langle X\rangle, ш_{\varphi}, 1_{X^*}, \Delta_{conc}, \varepsilon) \tag{30}$$

(ii) *The bialgebra (eq. 30) is, in fact, a Hopf Algebra.*

[a] When $|X| \geq 2$, noncocommutative.

The I_+ technology.

6. We have the following theorem
   Theorem. — Let $\mathcal{B} = (\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \varepsilon)$ be a bialgebra, then
   A) $\mathcal{B} = \ker(\epsilon) \oplus \mathbf{k}.1_{\mathcal{B}}$ and the projectors are
   i) $h \mapsto I_+(h) = h - \epsilon(h).1_{\mathcal{B}}$ on $\ker(\epsilon) = \mathcal{B}_+$.
   ii) $h \mapsto \mathrm{e}(h) = \epsilon(h).1_{\mathcal{B}}$ on $\mathbf{k}.1_{\mathcal{B}}$.
   B) If I_+ is locally nilpotent i.e.

$$(\forall b \in \mathcal{B})(\exists N \geq 0)(\forall n \geq N)({I_+^*}^n(b) = 0) \tag{31}$$

   then $\mathcal{B}$ is a Hopf algebra.
   C) (CQMM) If $\mathbb{Q} \subset \mathbf{k}$ and Δ is cocommutative, then TFAE
   i) $\mathcal{B}$ is an enveloping bialgebra.
   ii) $\mathcal{B} = \mathcal{U}(Prim(\mathcal{B}))$.
   iii) $\Delta_+ = {I_+^{\otimes}}^2 \circ \Delta$ is locally nilpotent.
   iv) I_+ is locally nilpotent.

CQMM: examples and counterexamples.

7. Let $\mathbf{k}$ be a ring, S be a subsemigroup of $\mathbb{N}$ and, for $s \in S$, $\Delta_{\uplus}(y_s) := \sum_{p+q=s} y_p \otimes y_q$, then

$$\mathcal{B} = \mathcal{B}_{\uplus} = (\mathbf{k}\langle Y \rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\uplus}, \epsilon) \tag{32}$$

is a bialgebra.

   i) If $S = \mathbb{N}_{\geq 1}$ (classical stuffle) and $\mathbf{k} = \mathbb{Z}$ $\mathcal{B}_{\uplus}$ is not an enveloping algebra.
   ii) With $S = \mathbb{N}$ and even $\mathbf{k} = \mathbb{Q}$ (we called this alphabet Y_0 in the Ph. D's), $\mathcal{B}_{\uplus}$ is not even a Hopf algebra.

8. **Remarks.** —
   i) (Weak form of the CQMM) With $\mathbb{Q} \subset \mathbf{k}$ and $\mathcal{B}$, connected, graded and cocommutative.
   Rq. — This, strictly weaker, form doesn't cover classical enveloping algebras as $\mathcal{U}(sl_2(\mathbf{k}))$.
   ii) In the equivalent conditions of CQMM, $\log(I) = \log(e + I_+)$ is the π_1 projector $\mathcal{B} \to Prim(\mathcal{B})$.

Enveloping algebras in context.

1. Let $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ be two categories and $F : \mathcal{C}_{right} \to \mathcal{C}_{left}$ a (covariant) functor between them

$$\begin{tikzcd}
\mathcal{C}_{left} & \mathcal{C}_{right} \arrow[l, dotted, "F"'] \\
U \arrow[r, dashed, "f"] \arrow[dr, dashed, "j_U"'] & V \\
 & Free(U) \arrow[u, "\hat{f}"']
\end{tikzcd}$$

Figure: A solution of the universal problem w.r.t. the functor F is the datum, for each $U \in \mathcal{C}_{left}$, of a pair $(j_U, Free(U))$ (with $j_U \in Hom(U, F[Free(U)])$, $Free(U) \in \mathcal{C}_{right}$).
$$\big(\forall f \in Hom(U, F[V])\big)\big(\exists!\, \hat{f} \in Hom(Free(U), V)\big)\big(F(\hat{f}) \circ j_U = f\big)$$

2. In the case of enveloping algebras $\mathcal{C}_{left} = \mathbf{k} - \mathbf{Lie}$, $\mathcal{C}_{right} = \mathbf{k} - \mathbf{AAU}$ and $F(\mathcal{A})$ is the algebra $\mathcal{A}$ endowed with the bracket $[x, y] = xy - yx$ thus a Lie algebra.

Limiting processes and topologies

1. We have seen last time some limiting processes (like Riemann integral and Lebesgues y-axis sampling) which are not reducible to sequences, (we will return to this point later on).
2. In order to understand deeply and master our calculations with group-like series (of all sorts not only for the co-shuffle coproduct), we have to deal with closed subgroups of the Magnus group.
3. Let us first examine and analyse some simple limits of sequences of series.
4. We first address the following identity

$$\lim_{n\to+\infty}(1+\frac{z}{n})^n = e^z \tag{33}$$

Which can be considered within the formal realm (i.e. LHS, for each n, within $\mathbb{C}\langle z\rangle = \mathbb{C}[z]$ and RHS within $\mathbb{C}\langle\langle z\rangle\rangle = \mathbb{C}[[z]]$) or in $\mathcal{H}(\mathbb{C})$ with compact convergence.

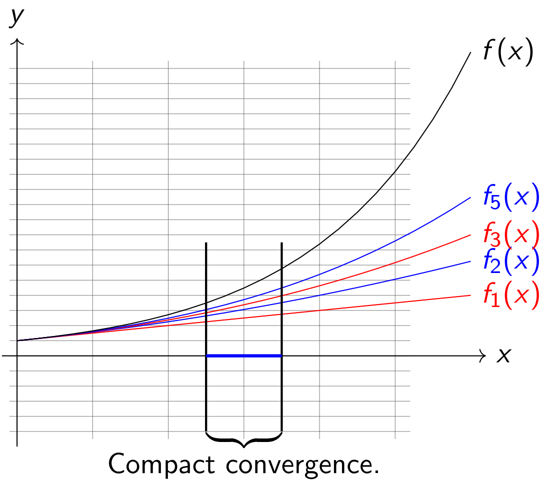


Figure: The one-parameter group $f(x) = \mathrm{e}^{\frac{x}{2}}$ as the limit of $f_n(x) = (1 + x/(2n))^n$.

Limiting processes and topologies/2

5. In fact, a variant of (33)[a] was used by Montgomery and Zippin to solve Hilbert's fifth problem [**?**].

6. (Informal) definition:[b] A one-parameter group, is a correspondence G to some group such that

$$G(t_1 + t_2) = G(t_1)G(t_2)$$

7. In fact, we are interested in creating a new theory of

   1. Paths drawn on groups of series
   2. One-parameter groups on infinite-dimensional Lie groups of series and their combinatorics.
   3. We use an application to stuffle identity, introducing a "Holomorphic functional calculus" [22] in order to get and prove non-trivial identities within Hausdorff groups.

[a] In fact, the construction of one-parameter groups as limits of this kind.

[b] Informal, means here "at the level of general idea".

Every path drawn on the group is a solution of

$$y'(t) = m(t)y(t)$$

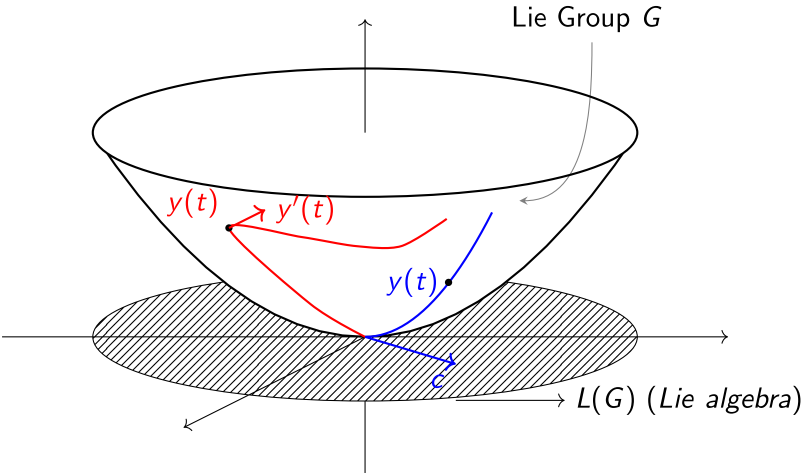


Figure: For one-parameter groups $y'(t)y(t)^{-1} = c$ is constant.

An identity in the stuffle algebra

8. We begin by an application on the Hausdorff group of a particular bialgebra. Here, with $Y = \{y_i\}_{i \geq 1}$

$$\mathcal{B} = \mathcal{B}_{\uplus} = (\underbrace{\mathcal{C}\langle Y \rangle, \mathtt{conc}, 1_{Y^*}}_{\text{algebra part}}, \Delta_{\uplus}, \epsilon) \tag{34}$$

and we first establish an identity within the stuffle algebra, taking "stars of the plane" as arguments.

$$(\sum_{i \geq 1} \alpha_i\, y_i)^* \uplus (\sum_{j \geq 1} \beta_j\, y_j)^* = (\sum_{i \geq 1} \alpha_i\, y_i + \sum_{j \geq 1} \beta_j\, y_j + \sum_{i,j \geq 1} \alpha_i \beta_j\, y_{i+j})^* \tag{35}$$

As the alphabet is infinite, we use here homogeneous series of degree one as $\sum_{i \geq 1} \alpha_i\, y_i$. These sums are not necessarily finite (they are, in general, a series) but can be so. Series like this form the vector space $\mathbb{C}^Y$ (called by Pr. Schützenberger "the plane of letters"), noted, in our works, $\widehat{\mathbb{C}.Y}$ as it is the completion of $\mathbb{C}.Y = \mathbb{C}^{(Y)}$ for some topology.

An identity in the stuffle algebra/2: Generalities

9. In fact, identity (35) describes completely the composition of characters (i.e. the composition within $\Xi(\mathcal{B})$). In fact $\mathcal{B}_{\uplus}$ (see its elements in eq. 34) is a conc-bialgebra and conc-characters are exactly "stars of the plane" i.e., for generic $\mathcal{X}$, of the form $(\sum_{x\in\mathcal{X}} \alpha_x\, x)^*$.

10. We recall that $\Delta_{\uplus}(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{\substack{p,q\geq 1\\ p+q=n}} y_p \otimes y_q$.

11. In fact this comultiplication is a particular case of $\Delta_{Ш_\varphi}$ comultiplications which read, for each letter $x \in \mathcal{X}$ (see [?]),

$$\Delta_{Ш_\varphi}(x) = x \otimes 1 + 1 \otimes x + \sum_{y,z\in\mathcal{X}} \gamma_{x}^{y,z}\, y \otimes z \tag{36}$$

where the tensor $\gamma_x^{y,z}$ is locally finite in x.

12. For these conc-bialgebras, we have in general

$$(\sum_{y\in\mathcal{X}} \alpha_y\, y)^* \,Ш_\varphi (\sum_{z\in\mathcal{X}} \beta_z\, z)^* = (\sum_{y\in\mathcal{X}} \alpha_y\, y + \sum_{z\in\mathcal{X}} \beta_z\, z + \sum_{x,y,z\in\mathcal{X}} \alpha_y \beta_z\, \gamma_x^{y,z} x)^* \tag{37}$$

An identity in the stuffle algebra/3: Generalities

13. One proof of (37) rests on the fact that the algebra is generated by $\mathcal{X}$ and, then, we have just, knowing the form of the LHS-RHS, to test equality on letters. Let us recall some definitions and properties (**k** is a commutative ring)

    1. Let $\mathcal{B} = (\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \epsilon)$ be a bialgebra.
    2. We call $\Xi(\mathcal{B})$ the set of characters of $(\mathcal{B}, , \mu, 1_{\mathcal{B}})$ (with values in **k**)
    3. When $\mathcal{C}$ is another **k**-algebra, we will note $\Xi(\mathcal{B}; \mathcal{C})$, the set of characters of $\mathcal{B}$ with values in $\mathcal{C}$.[a]

14. One can show that, if $\mathcal{C}$ is commutative, characters compose through convolution. Indeed, the dual $\mathcal{B}^\vee$ (now $\mathcal{C} = \mathbf{k}$) is an algebra under ${}^t\Delta$ (which will be noted $\circledast$) and $\Xi(\mathcal{B}) \subset \mathcal{B}^\vee$ is closed under $\circledast$.

---

[a]This set is none other than the Hom-set of the algebras, i.e. we have truly

$$\Xi(\mathcal{B}; \mathcal{C}) = Hom_{\mathbf{k}-\mathbf{AAU}}(\mathcal{B}, \mathcal{C})$$

but the point of view is commpletely different.

An exercise about these generalities

15. Let $\mathbf{k}$ be a commutative ring and $\mathcal{B} = (\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \epsilon)$ be a $\mathbf{k}$-bialgebra. As $\Delta : \ \mathcal{B} \longrightarrow \mathcal{B} \otimes \mathcal{B}$, we have ${}^t\Delta : (\mathcal{B} \otimes \mathcal{B})^{\vee} \longrightarrow \mathcal{B}^{\vee}$
16. (Q1) Explain the arrow

$$can : \ \mathcal{B}^{\vee} \otimes \mathcal{B}^{\vee} \ \longrightarrow \ (\mathcal{B} \otimes \mathcal{B})^{\vee} \tag{38}$$

and prove that ${}^t\Delta \circ can$ is a law of $\mathbf{k} - \mathbf{AAU}$ in $\mathcal{B}^{\vee}$ (we will note this law $\circledast$).

17. (Q2) i) Let $\mathcal{C}$ be a $\mathbf{k} - \mathbf{CAAU}$, prove that $\Xi(\mathcal{B})$ is a submonoid of $(\mathcal{B}^{\vee}, \circledast, \epsilon)$.
ii) Extend these results to $\Xi(\mathcal{B}; \mathcal{C})$ (where $\mathcal{C}$ is an object of $\mathbf{k} - \mathbf{CAAU}$).
18. (Q3) i) For $t \in \mathbb{C}$, compute $(2ty_1 + t^2 y_2)^*$ under the form of an exponential.
ii) Recall that "Stars of the plane" are $\mathtt{conc}$-characters and prove that, for $t \neq 0$, $(y_1^*, (2ty_1 + t^2 y_2)^*, y_3^*)$ are algebraically independent over $(\mathbb{C}\langle Y \rangle, \,\sqcup\!\sqcup\,, 1_{Y^*})$ within $(\mathbb{C}\langle\!\langle Y \rangle\!\rangle, \,\sqcup\!\sqcup\,, 1_{Y^*})$.
iii) More generally, prove that, if $Q_i \in \widehat{\mathbb{C}.Y}$ are $\mathbb{Z}$-linearly independent, then $(Q_i^*)_{i \in I}$ are algebraically independent.

Exercise (cont'd)

19. Before proving the (very hard) question (iii) of exercise 18 above let us give a bit of a categorical motivation.

20. $\mathcal{H}(\Omega)$ is a $\mathbb{C}$-vector space, in fact a $\mathbb{C}-\mathbf{CAAU}$ (and hence all derived substructures: monoid and the like). Then, if one has a correpondence (a set-theoretical map)

$$\Phi_{set} : \mathcal{X} \longrightarrow \mathcal{H}(\Omega) \tag{39}$$

(be it for "inputs" or everything else, arbitrary) one can extend it to $\mathbb{C}\langle\mathcal{X}\rangle$ as we do for $\alpha_{z_0}^{z}$, Θ, ... One gets at once an extension

$$\Phi_{\mathbb{C}-\mathbf{AAU}} : \mathbb{C}\langle\mathcal{X}\rangle \longrightarrow \mathcal{H}(\Omega) \tag{40}$$

21. The question will be addressed next time will be to extend (40) to (certain) series.

22. On the RHS of (40), we have a space with a topology (apparently, the only reasonable one, see [26]). On the LHS, there are several topologies.

An algebraic one-parameter group for stuffles/1

23. (Holomorphic functional calculus [22]) Let $S \in \mathbb{C}_+\langle\langle Y \rangle\rangle$ (sometimes called "a proper series") and $T = \sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$, we first remark that $(a_n S^{\uplus\, n})_{n\geq 0}$ is "summable" (see definition below, equation (41) and use the weight).

Definition

A family of series $(S_i)_{i\in I}$ in $\mathbf{k}\langle\langle \mathcal{X} \rangle\rangle$ is said *summable* if, for all $w \in \mathcal{X}^*$, the map $i \mapsto \langle S_i \mid w \rangle$ is finitely supported. In this case the sum of the family is defined by

$$\sum_{i\in I}(S_i) := \sum_{w\in\mathcal{X}^*} \sum_{i\in I} \langle S_i \mid w \rangle\, w \tag{41}$$

24. For $T \in \mathbb{C}[[z]]$ and $S \in \mathbb{C}_+\langle\langle Y \rangle\rangle$, we note

$$T_{\uplus}(S) := \sum_{n\geq 0} \langle T \mid z^n \rangle\, S^{\uplus\, n} \tag{42}$$

An algebraic one-parameter group for stuffles/2

25. For $S \in \mathbb{C}_+\langle\langle Y \rangle\rangle$, we have

$$\log_{\uplus}(1_{Y^*} + S) \ \ \exp_{\uplus}(S) - 1_{Y^*} \text{ belong to } \mathbb{C}_+\langle\langle Y \rangle\rangle \text{ and} \tag{43}$$

$$\exp_{\uplus}(\log_{\uplus}(1_{Y^*} + S)) = 1_{Y^*} + S \ \ \log_{\uplus}(\exp_{\uplus}(S)) = S \tag{44}$$

26. (Commutation and polynomial type coefficients) For $S, T \in \mathbb{C}_+\langle\langle Y \rangle\rangle$ and $P(z) \in \mathbb{C}[z]$, we have

$$\exp_{\uplus}(S + T) = \exp_{\uplus}(S) \uplus \exp_{\uplus}(T) \text{ and} \tag{45}$$

$$\exp_{\uplus}(P(z).S) \in \mathbb{C}[z]\langle\langle Y \rangle\rangle \ ; \tag{46}$$

$$\frac{d}{dz}(\exp_{\uplus}(P(z).S)) = (P'(z).S) \uplus \exp_{\uplus}(P(z).S) \tag{47}$$

An algebraic one-parameter group for stuffles/3

27. Now, we code "the plane" by Umbral calculus.
28. Let x be an auxiliary letter, The map

$$\pi_Y^{Umbra} : \sum_{n\geq 1} \alpha_n\, x^n \mapsto \sum_{n\geq 1} \alpha_n\, y_n \tag{48}$$

from $\mathbb{C}_+[[x]]$ to $\widehat{\mathbb{C}.Y}$ is linear and bijective. We will call π_x^{Umbra} its inverse.

29. For $S, T \in \mathbb{C}_+[[x]]$, one can show that

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^* \tag{49}$$

30. Therefore, for $z \in \mathbb{C}$ and $T \in \mathbb{C}_+[[x]]$, one sets

$$G(z) = (\pi_Y^{Umbra}(e^{z.T}-1))^* \tag{50}$$

An algebraic one-parameter group for stuffles/4

31. From (49), (47) and (35) one gets, for $z_1, z_2 \in \mathbb{C}$,

$$G(z_1 + z_2) = G(z_1) \uplus G(z_2) \ ; \ \ G(0) = 1_{Y^*} \tag{51}$$

(then G can truly be called a "stuffle one parameter group").

32. We check that

$$\frac{d}{dz}(G(z)) = (\pi_Y^{Umbra}(T)) \uplus G(z) \tag{52}$$

and deduce that

$$G(z) = e_{\uplus}^{z.\pi_Y^{Umbra}(T)} \tag{53}$$

33. What precedes shows us that, for each $P = \sum_{i \geq 1} \langle P \mid y_i \rangle \, y_i \in \widehat{\mathbb{C}.Y}$

$$\log_{\uplus}(P^*) = \pi_Y^{Umbra}(\log(1 + \pi_x^{Umbra}(P))) \tag{54}$$

An algebraic one-parameter group for stuffles/5

34 In particular, using (54), we show that

$$(ty_k)^* \quad = \quad \exp_{\uplus}\Big(\sum_{n\geq 1}\frac{(-1)^{n-1}t^n y_{nk}}{n}\Big) \tag{55}$$

Limiting processes and topologies/3

35 Our first examples are taken in $\mathbb{C}[[z]] = \mathbb{C}\langle\langle z\rangle\rangle$.

36 First, we return to S^* (S is without constant term) and $(1+\frac{z}{n})^n$.

37 In the first case, calling $\omega(S)$ the minimal length of *supp*(S) (and still supposing $\langle S \mid 1_{\mathcal{X}^*}\rangle = 0$) we have $\omega(S^n) \geq n$ and then $(S^n)_{n\geq 0}$ is summable.

38 In the second one, one has

$$(1+\frac{z}{n})^n = 1 + z + \frac{(n)(n-1)}{n^2}z^2 + \ldots = 1 + z + \frac{(n-1)}{n}z^2 + \ldots \quad (56)$$

the series of differences $T_n = (1+\frac{z}{n+1})^{n+1} - (1+\frac{z}{n})^n$ is NOT summable as $T_n = \frac{1}{n(n+1)}z^2 + \ldots$ and then for all $n \in \mathbb{N}$, $\omega(T^n) = 2$. What happens in fact is that, for all $N \in \mathbb{N}$,

$$\lim_{n\to\infty}\langle (1+\tfrac{z}{n})^n \mid z^N\rangle = \frac{1}{N!}$$

so that, even if the series of differences is not summable, the limit exists. This term-by-term topology (which is the product topology) is called "Treves Topology" in [14] (see [38] Ch10 Example III).

Concluding remarks.

1. Extending the domain of polylogarithms to (some) rational series permits the projection of rational identities. Such as

$$(\alpha x)^* \sqcup\!\sqcup (\beta y)^* = (\alpha x + \beta y)^*$$

2. The theory developed here allows to pursue, for the Harmonic sums, this investigation such as

$$(\alpha y_i)^* \uplus (\beta y_j)^* = (\alpha y_i + \beta y_j + \alpha\beta y_{i+j})^*$$

3. We have, on the left, spaces equipped with Krull ultrametric convergence and a nice setting on the (topological) Magnus and Hausdorff groups. On the right, we have adapted domain theories with identities between polylogarithms and harmonic sums.
4. We have discussed general CQMM and its consequences for MRS.

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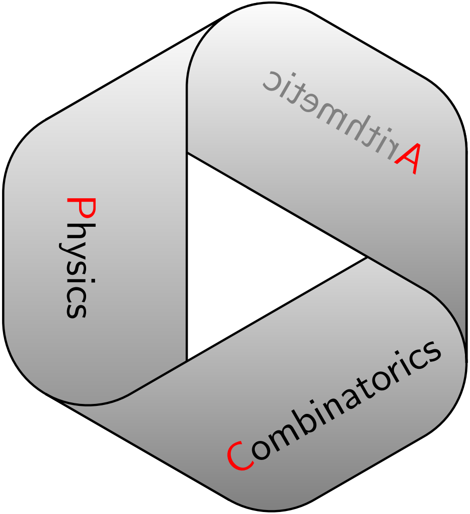

Figure: ... and a lot of (machine) computations.

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ANNALES DE L'INSTITUT HENRI POINCARÉ D

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